

LOW THRESHOLD BOOTSTRAP PERCOLATION ON THE HAMMING TORUS

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Bootstrap percolation first appeared in a paper by Chalupa et al [7] as a model for ferromagnetism. Adler and Lev [1] provide a wonderful introduction to the subject.

The process takes place on a graph $G = (V, E)$ with vertex set V and edge set E and depends on a parameter θ which we call the *threshold*. Each vertex in the graph can be in one of two states, either open or closed. At each subsequent step a vertex becomes open if at least θ of its neighbors are open. Once open, a vertex remains open for all eternity.

We now formalize the evolution of increasing configurations. Let $\omega_t \in \{0, 1\}^V$ denote the configuration of the vertices at time $t \geq 0$. If a vertex v is open at step t we say $\omega_t(v) = 1$ and similarly if the vertex is closed at time t , $\omega_t(v) = 0$. For bootstrap percolation with threshold θ , ω_t evolves as follows for $t \geq 0$:

$$(1) \quad \omega_{t+1}(v) = \begin{cases} 1, & \omega_t(v) = 1 \text{ or } \sum_{v' \sim v} \omega_t(v') \geq \theta \\ 0, & \text{otherwise} \end{cases}$$

where $v' \sim v$ if there is an edge in E connecting v and v' .

In this paper we will assume $\mathbb{P}(\omega_0(v) = 1) = p$ independently for each v . Given some initial configuration, we can ask what the evolved configuration will look like after some time. In particular we care about the steady state, ω_∞ . Typically this is viewed probabilistically. Given a distribution on ω_0 what can we say about ω_∞ ?

The first rigorous results came from van Enter [15] and later Schonmann [14]. They showed that no non-trivial phase transition on the infinite lattice \mathbb{Z}^d with edges connecting each vertex to its $2d$ nearest neighbors. For $\theta \leq d$, every point eventually becomes open with probability 1 if $p > 0$. If $\theta > d$ then everything becomes completely open with positive probability only if $p = 1$.

The next big step in the history of bootstrap percolation was to view the process on a family of finite graphs $\mathcal{G} = \{G_n = (V_n, E_n)\}$ where the probability that a vertex is initially open is given by a function of n , $p = p(n)$. As each graph is finite, for any increasing event A , $f_A(p) := \mathbb{P}_p(A)$ is an increasing polynomial in p with $f_A(0) = 0$ and $f_A(1) = 1$. By continuity, for each $\alpha \in [0, 1]$ there is some $p_\alpha = p_\alpha(n)$, such that $f_A(p_\alpha) = \alpha$. As is customary, we let p_c denote the critical probability $p_{1/2}$.

We say there is a sharp phase transition for an increasing event A if a small perturbation from the critical probability drastically changes the probability of A . More formally

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the phase transition is sharp if for any $\epsilon \in (0, 1)$,

$$p_{1-\epsilon} - p_\epsilon = o(p_c).$$

Friedgut and Kalai [8] investigate this phenomenon in some generality.

Aizenman and Lebowitz [2] showed for the event $A = \{\omega_\infty \equiv \mathbf{1}\}$ on the finite d dimensional grid, $[n]^d$, and threshold $\theta = 2$, there exists constants c_1, c_2 such that $c_1 < (\log n)^{d-1} p_c < c_2$. Moreover, they show that the phase transition is sharp.

In a widely celebrated paper Holroyd [11] showed that for $d = \theta = 2$

$$p_c \sim \pi^2 / 18 \log n.$$

Later this result was expanded on by Holroyd, Liggett, and Romik [12] to $d = 2, \theta = k+1$ where the neighborhood of a vertex is the k closest vertices in each of the cardinal directions. They show $p_c \sim \pi^2 / (3(k+2)(k+1) \log n)$ for this graph. These types of results have been extended to higher dimensions by [4], hypercubes [3], random graphs [5], and more geometric settings [6]. This is a very active area of research.

Our graph of interest is the d -dimensional Hamming torus. The Hamming torus has the same vertex set as the torus, $V = [n]^d$, but the edge set is modified to connect every vertex that can be connected with a straight path in a coordinate direction. That is

$$E := \{(v, w) : v \text{ differs from } w \text{ in exactly one coordinate}\}.$$

Gravner et al. [9] introduced the study of bootstrap percolation on the Hamming torus. For general thresholds $\theta \geq 2$ they investigate the critical probability, $p_c = p_c(\theta, d)$. They also consider finer structure, which we now introduce.

Definition 0.1. A subset $V \subset [n]^d$ is a **subtorus** if there exists a set of indices $I(V)$ and constants $\{\alpha_l\}_{l \in I(V)}$ such that $v \in V$ if and only if for all $l \in I(V)$, $v_l = \alpha_l$. For fixed d , we say V has dimension i if $|I(V)| = d - i$ and denote by \mathcal{F}_i the collection of all such subtori.

For $0 \leq i \leq d$, they study the events

$$\mathcal{C}_i = \{\exists V \in \mathcal{F}_i \text{ s.t. } \omega_\infty|_V \equiv \mathbf{1}\}.$$

Let $p_c(\theta, i, d)$ denote the critical probability for the event \mathcal{C}_i . Gravner et al. show for $d = 2$ and any $\theta \geq 2$ that $p_c(\theta, 1, 2) = p_c(\theta, 2, 2)$ and

$$\mathbb{P}_{p_c}(\{\omega_\infty \neq \omega_0\} \setminus \{\mathcal{C}_d\}) = o(1).$$

For $d = \theta = 3$, and $p = an^{-2}$ they compute a precise limiting value of $\mathbb{P}_p(\mathcal{C}_3)$ that varies continuously from 0 to 1 as a increases from 0 to infinity. In particular, the transition is not sharp.

For an increasing event A , we say the γ is a critical exponent for A if for any $\epsilon > 0$

$$\mathbb{P}_p(A) \rightarrow \begin{cases} 1, & p > n^{-\gamma+\epsilon} \\ 0, & p < n^{-\gamma-\epsilon} \end{cases}.$$

For larger d and θ they prove upper and lower bounds on the critical exponent for \mathcal{C}_2 provided it exists. For large enough d and θ they show this is different than the critical exponent of \mathcal{C}_1 .

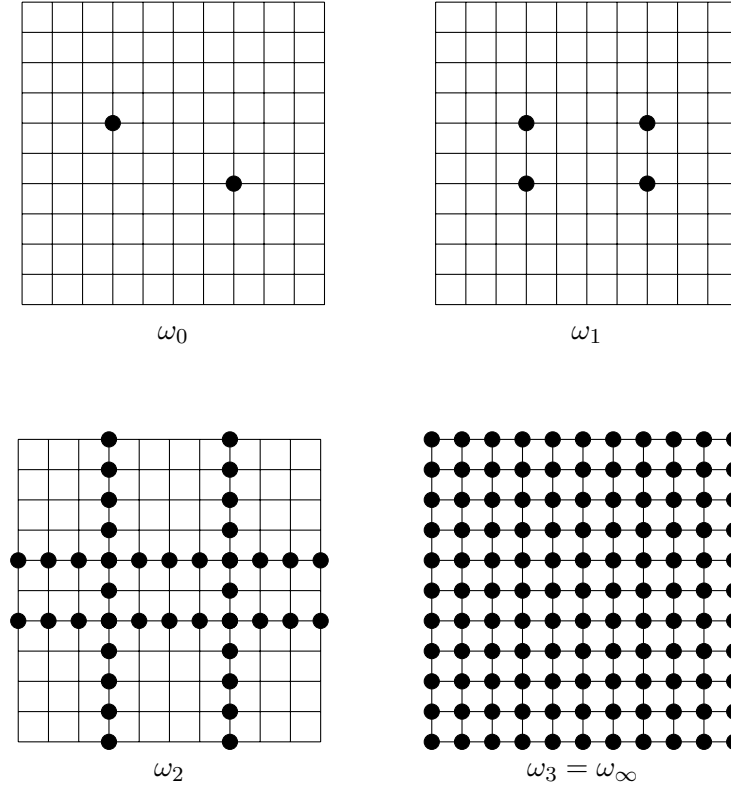


FIGURE 1. The bootstrap percolation process with threshold $\theta = 2$ starting with two non-colinear open nodes.

We consider the case $\theta = 2$ and $d > 2$. The case where $\theta \geq 2$ and $d = 2$ is well understood. (See Figure 1 for a picture of the process with $d = \theta = 2$). For fixed $d > 2$, define

$$J = J_d = \max\{j : j(j+1) < d\}.$$

We show that the critical exponents for $\mathcal{C}_2, \mathcal{C}_4, \dots, \mathcal{C}_{2J}$ are distinct. We also show for every j such that $2 \leq 2j \leq d$, the critical exponent for \mathcal{C}_{2j} and \mathcal{C}_{2j-1} are the same and

$$\mathbb{P}_p(\mathcal{C}_{2j} \setminus \mathcal{C}_{2j-1}) \rightarrow 0.$$

If $(J+1)(J+2) > d$ then we have $\mathbb{P}(\mathcal{C}_{2J} \setminus \mathcal{C}_d) \rightarrow 0$. Whereas if $(J+1)(J+2) = d$ then \mathcal{C}_{2J} and \mathcal{C}_d have the same critical exponent, but $\mathbb{P}_p(\mathcal{C}_{2J} \setminus \mathcal{C}_d)$ has a non-zero limit.

After we have determined the critical exponent for these events, we will give a precise description of the asymptotics of $p_c(2, i, d)$. Unlike the threshold functions for the grid $[n]^d$ found in Bollobás [4], $p_c(2, i, d)$ is not sharp. This is due the large neighborhood size relative to the total number of vertices.

1. STATEMENTS

First, we need a few definitions. We will identify ω_t with the set $\{v : \omega_t(v) = 1\}$.

Definition 1.1. For a set of nodes, S , we define their **span**, $\langle S \rangle$, to be the set ω_∞ of eventually occupied points starting from $\omega_0 = S$. We say V is **internally spanned** by S if $V = \langle S \cap V \rangle$.

For random ω_0 we consider the following events:

- $\mathcal{I}_V = \{\omega_0 \text{ internally spans } V\}$,
- $\mathcal{I}_i = \{\exists V \in \mathcal{F}_i \text{ s.t. } \mathcal{I}_V \text{ occurs}\} = \bigcup_{V \in \mathcal{F}_i} \mathcal{I}_V$,
- $\mathcal{C}_i = \{\exists V \in \mathcal{F}_i \text{ s.t. } \omega_\infty|_V \equiv \mathbf{1}\}$.

Note the slight difference in the definitions of \mathcal{I}_i and \mathcal{C}_i . For \mathcal{C}_i the only thing that matters is the final state ω_∞ where for \mathcal{I}_i it is important how one gets to ω_∞ .

For the remainder of this paper we drop the parameter θ as it will always be 2. Throughout the paper we will assume $d > 2$ as that case was answered completely for all θ in [9]. For $d > 2$, and $0 \leq i \leq d$ denote the threshold functions of \mathcal{I}_i and \mathcal{C}_i by $p_{\mathcal{I}}(i, d)$ and $p_{\mathcal{C}}(i, d)$ respectively. Much of the work in this paper is in finding bounds for the threshold function for \mathcal{I}_i . Then we show that $p_{\mathcal{C}}(i, d)$ will have the same asymptotic behavior as $p_{\mathcal{I}}(i, d)$ when i is even.

Now we are in a position to state our main results. To shorten the statements of the following theorems we define

$$\lambda(j, d, a) := \binom{d}{2j} (2j)! 2^{-j-1} a^{j+1}.$$

Theorem 1.1. Fix $d > 2$ and $j \leq J$, and let $p = an^{-d/(j+1)-j}$, $\lambda_j = \lambda(j, d, a)$. Then

$$(2) \quad \mathbb{P}_p(\mathcal{I}_{2j}) \rightarrow 1 - e^{-\lambda_j},$$

and

$$(3) \quad \mathbb{P}_p(\mathcal{C}_{2j} \setminus \mathcal{I}_{2j}) \rightarrow 0.$$

In fact to prove Theorem 1.1 part 2 we prove a stronger result on Poisson convergence by an application of the Chen-Stein method [13]. For two non-negative integer valued random variables Y and Z the total variation is defined as

$$d_{TV} = \frac{1}{2} \sum_{k=0}^{\infty} |\mathbb{P}(Y = k) - \mathbb{P}(Z = k)|.$$

Theorem 1.2. Fix $d > 2$. Let $j \leq J$, $p = an^{-d/(j+1)-j}$, and $\lambda_j = \lambda(j, d, a)$. Let Y_j denote the number of subtori $V \in \mathcal{F}_{2j}$ such that \mathcal{I}_V occurs, and let Z_j denote a $\text{Poisson}(\lambda_j)$ random variable. Then

$$\lim_{n \rightarrow \infty} d_{TV}(Y_j, Z_j) \rightarrow 0.$$

The precision given by Theorem 1.2 leads to the following results:

Theorem 1.3. Fix $d > 2$ such that $d < (J+1)(J+2)$ and let $p = an^{-d/(J+1)-J}$. Then

$$\mathbb{P}_p(\mathcal{I}_{2J} \setminus \mathcal{I}_d) \rightarrow 0.$$

Theorem 1.4. *Fix $J \geq 1$ and let $d = (J+1)(J+2)$, $p = an^{-2J-2}$, and $\lambda_J = \lambda(J, d, a)$. There exists positive constants in $0 < c_1 < c_2 < 1 - e^{-\lambda_J}$ such that for large enough n*

$$(4) \quad \mathbb{P}_p(\mathcal{I}_{2J} \setminus \mathcal{I}_{2J+2}) > c_1$$

$$(5) \quad \mathbb{P}_p(\mathcal{I}_{2J+2}) > c_2$$

and

$$(6) \quad \mathbb{P}_p(\mathcal{I}_{2J+2} \setminus \mathcal{I}_d) \rightarrow 0.$$

The following theorem highlights how $J = 1$ is different from higher J when $d = (J+1)(J+2)$.

Theorem 1.5. *Fix $J \geq 1$ and let $d = (J+1)(J+2)$ and $p = an^{-2J-2}$. If $J > 1$ then*

$$(7) \quad \mathbb{P}_p(\mathcal{I}_d \setminus \mathcal{I}_{2J+2}) \rightarrow 0,$$

whereas if $J = 1$, then there exists $c > 0$

$$(8) \quad \mathbb{P}_p(\mathcal{I}_6 \setminus \mathcal{I}_4) > c.$$

In Section 2, we prove lemmas that describe the evolution of ω_t when $\theta = 2$. In Section 3, we prove upper and lower bounds for the events \mathcal{C}_{2j} and \mathcal{I}_{2j} . In Section 4 we use the Chen-Stein method [13] to describe precisely the asymptotics of $p_c(i, d)$ and $\mathbb{P}(\mathcal{I}_{2J})$. In Section 5 we combine everything to prove our statements.

2. DETERMINISTIC RESULTS

We begin with the simplest case. Suppose $u \neq v$ are the only nodes which are initially open. Define the [Hamming] distance between the nodes as $\text{dis}(u, v) := \sum_{i=1}^d \mathbf{1}_{u_i \neq v_i}$, the number of coordinates where u and v differ. If $\text{dis}(u, v) > 2$ then no new nodes become open $\langle \{u, v\} \rangle = \{u, v\}$. If $\text{dis}(u, v) \leq 2$ then u and v must agree for all but at most 2 indices. Without loss of generality, we may assume that $u_i = v_i$ for $i > 2$.

Suppose first that $u_2 = v_2$ as well (i.e. $\text{dis}(u, v) = 1$), the line $\{(t, u_2, \dots), t \in [n]\}$ has two nodes initially open, and after one step every node in that line becomes open. Every node not on the line has at most one neighbor on the line, so growth stops.

If $\text{dis}(u, v) = 2$, then after one step the common neighbors of u and v , $u' = (u_1, v_2, \dots)$ and $v' = (v_1, u_2, \dots)$, become open. The nodes u and u' are two different open neighbors for every closed node in the line $\{(u_1, s, \dots) : s \in [n]\}$, so after two steps the entire line becomes open. The same is true for the lines containing both u and v' , both v and u' , and both v' and v . Once those lines are open every other node in the plane $\{(t, s, \dots) : (t, s) \in [n]^2\}$ has at least two (in fact four) open neighbors, so the entire plane becomes open. (See Figure 1)

Growth for higher dimension subtori is a bit more involved. First we generalize the distance function to subsets S_1, S_2 as follows,

$$\text{dis}(S_1, S_2) = \inf_{u \in S_1, v \in S_2} \text{dis}(u, v).$$

We will state and prove a few necessary lemmas. The key point is that growth continues only if there are two sets of open nodes within distance 2 of each other.

Lemma 2.1. *For $S \subset [n]^d$, let \overline{S} denote the smallest subtorus that contains S . If V is a subtorus and u is a node with $\text{dis}(V, u) \leq 2$ then*

$$\langle V \cup \{u\} \rangle = \overline{V \cup \{u\}}.$$

Proof. (By induction on $i = \dim(V)$) We have shown that the lemma holds if V has dimension 0 (is a single node). Suppose the lemma holds for all subtori W with $\dim(W) < i$. Let V be a subtorus with $\dim(V) = i$ and let u be a node with $\text{dis}(V, u) \leq 2$. Without loss of generality we assume the last $d - i$ coordinates are fixed, i.e. $I(V) = [i + 1, d]$. Without loss of generality we may also assume that

$$u \in \{(u_1, \dots, u_d) : u_l = \alpha_l(V) \text{ for } l > i + 2\}.$$

Let V_k denote the subtorus of V that fixes the k^{th} coordinate to the value u_k . Then V_k has dimension $i - 1$ and $\text{dis}(V_k, u) \leq 2$. By the induction hypothesis, $\langle V_k, u \rangle = \overline{V_k \cup \{u\}}$. For $a = (a_1, \dots, a_d) \in \overline{V \cup \{u\}}$, there are two neighbors

$$b = (u_1, a_2, \dots, a_d) \in \overline{V_1 \cup \{u\}}$$

and

$$c = (a_1, u_2, \dots, a_d) \in \overline{V_2 \cup \{u\}},$$

so a becomes open and we can conclude $\overline{V \cup \{u\}} \subseteq \langle V_1, V_2, u \rangle \subseteq \langle V, u \rangle$. Trivially $\langle V, u \rangle \subseteq \overline{V \cup \{u\}}$ so we have equality for the two sets. Moreover, if $u \notin V$ then $i + 1 \leq \dim(\overline{V \cup \{u\}}) \leq i + 2$. ■

Lemma 2.2. *If V, W are open subtori and $\text{dis}(V, W) \leq 2$ then $\langle V, W \rangle = \overline{V \cup W}$.*

Proof. This is a natural extension of Lemma 2.1. Trivially we have $\langle V, W \rangle \subseteq \overline{V \cup W}$. Let $V^0 = V$. We define V^l recursively. Let W^{l-1} denote the subset of W that satisfies $0 < \text{dis}(V^{l-1}, u) \leq 2$ for every $u \in W^{l-1}$. For $l > 0$ if $W \cap (V^{l-1})^c$ is non-empty there exists a $w_l \in W^{l-1}$. We then define $V^l = \langle V^{l-1}, w_l \rangle$ for some choice of w_l . By Lemma 2.1 this is the subtorus $\overline{V^{l-1} \cup \{w_l\}}$. Its dimension is strictly greater than $\dim(V^{l-1})$. If $W \cap (V^{l-1})^c$ is empty then $V^l = V^{l-1}$.

Since $\{V^l\}$ is an increasing sequence of subtori bounded by $\overline{\{V, W\}}$ it must stabilize to some subtorus V^m in a finite number of steps. By definition $V \subseteq V^m$, and more importantly, $W \cap (V^m)^c = \emptyset$ so $W \subseteq V^m$. Since $V^m = \langle V \cup \{w_1, \dots, w_m\} \rangle$ we also have that $V^m \subseteq \langle V, W \rangle$. Combining everything we get

$$\overline{V \cup W} \subseteq V^m \subseteq \langle V, W \rangle \subseteq \overline{V \cup W}$$

and the lemma holds. ■

Definition 2.1. *A subtorus V is **maximal** in $\langle S \rangle$ if no other subtorus in $\langle S \rangle$ contains V .*

The next two lemmas give conditions for when and how a subtorus is internally spanned.

Lemma 2.3. *For an initial configuration of open nodes S , let V be a maximal subtorus in $\langle S \rangle$. Then V is internally spanned with $V = \langle S \cap V \rangle$.*

Proof. Let $S_1 = S \cap V$ and $S_2 = S \setminus S_1$. If $\langle S_1 \rangle = V$ then we are done. Suppose that $\langle S_1 \rangle \neq V$. Since V eventually becomes open, there must be some node $u \in \langle S_2 \rangle$ such that $\text{dis}(\langle S_1 \rangle, u) \leq 2$, otherwise evolution would stop and V could not be contained in $\langle S \rangle$. In particular, there is a node $u \in \langle S_2 \rangle$ such that $u \notin V$ yet $\text{dis}(V, u) \leq 2$. By Lemma 2.2 the smallest subtorus that contains both u and V becomes open eventually. However V is maximal so no such u can exist and $\langle S_1 \rangle = V$. ■

Lemma 2.4. *Let S be a set of open nodes in $[n]^d$ with $V \subset \langle S \rangle$ a maximal open subtorus. There exist disjoint non-empty subsets $S_1, S_2 \subset S$ and subtori $V_1, V_2 \subset V$ with $\dim(V_1) \leq \dim(V_2) < \dim(V)$ such that $\langle S_1 \rangle = V_1$, $\langle S_2 \rangle = V_2$, and $\langle S_1 \cup S_2 \rangle = V$.*

Proof. V is maximal so we may assume $\langle S \rangle = V$. Consider the sequence of nested collections of subtori contained in $\langle S \rangle$,

$$\{W_i^0\} \subset \{W_i^1\} \subset \cdots \subset \{W_i^k\} \subset V$$

where $S = \{W_i^0\}$ and $\{W_i^{k+1}\}$ is formed by finding two subtori $W_{i_1}^k$ and $W_{i_2}^k$ within Hamming distance 2 of each other and setting $W_{i_1}^{k+1} = \langle W_{i_1}^k \cup W_{i_2}^k \rangle$ and reindexing the others appropriately. Since S is finite, eventually we will have two subtori $W_{i_1}^k, W_{i_2}^k \neq V$ such that $\langle W_{i_1}^k \cup W_{i_2}^k \rangle = V$. Each $W_{i_l}^k$ had a set S_l such that $\langle S_l \rangle = W_{i_l}^k$ for $l = 1, 2$. ■

3. CRITICAL PROBABILITY

To find the asymptotics of $p_c(2j, d)$, we will first prove upper and lower bounds for the exponent of $p_{\mathcal{I}}(2j, d)$. Since $\mathcal{I}_{2j} \subset \mathcal{C}_{2j}$ any upper bound for $p_{\mathcal{I}}(2j, d)$ will hold for $p_c(2j, d)$. With a little more work, we then prove the lower bound for the exponent of $p_{\mathcal{I}}(2j, d)$ will also be a lower bound for the exponent of $p_c(2j, d)$.

For odd dimension subtori we will show that $\mathbb{P}_p(\mathcal{I}_{2j-1}) \leq (1 + o(1))\mathbb{P}_p(\mathcal{I}_{2j})$ hence asymptotically $p_c(2j-1, d) \sim p_c(2j, d)$. This is apparent in the case of a line and a plane. For a line to be internally spanned, two nodes need to be co-linear, whereas for a plane to be internally spanned, two nodes only need to be co-planar.

3.1. Upper Bound. For fixed d and p , the probability of \mathcal{I}_V is identical for $V \in \mathcal{F}_i$. We then denote for any particular $V \in \mathcal{F}_i$

$$(9) \quad M_i := \mathbb{P}_p(\mathcal{I}_V).$$

Lemma 3.1. *For any $\epsilon > 0$ let $p < n^{-2J-\epsilon}$. For $1 \leq i \leq J$, there exists constant $c_d > 0$ such that for every $V \in \mathcal{F}_{2i}$ and n large*

$$\mathbb{P}_p(\mathcal{I}_V) = M_{2i} \geq (2i)! 2^{-i-1} n^{i(i+3)} p^{i+1} (1 - n^{-c_d}).$$

Proof of Lemma 3.1. Let V be a subtorus with dimension $2i$. Suppose we have a collection of distinct nodes $S = \{v_1, \dots, v_{i+1}\} \subset V$ such that $\langle \{v_1, \dots, v_{i+1}\} \rangle = V$. The probability that only these nodes are open is exactly $p^{i+1}(1-p)^{n^{2i}-i-1}$. Let \mathcal{L}_V be the set of all such collections. Since $p < n^{-2J-\epsilon}$ and $i \leq J$ there exists constant $\beta_d > 0$ such that $(1-p)^{n^{2i}-i-1} \geq (1-n^{-\beta_d})$. Then

$$(10) \quad M_{2i} \geq \sum_{\mathcal{L}_V} p^{i+1} (1-p)^{n^{2i}-i-1} \geq |\mathcal{L}_V| p^{i+1} (1-n^{-\beta_d}).$$

We call a ordered collection, $S = \{v_1, \dots, v_{i+1}\}$ *perfect* in V if the following are satisfied:

- $\langle S \rangle = V$,
- for $1 \leq i_1 < i_2 \leq i+1$, $\text{dis}(v_{i_1}, v_{i_2}) = 2(i_2 - 1)$,
- and $v_1 < v_2$ in lexicographical ordering.

For $i' \leq i$, the subcollection $S_{i'} = \{v_1, \dots, v_{i'+1}\}$ is also perfect in $\langle S_{i'} \rangle = V'$ and $\dim(V') = 2i'$. Note that a non-trivial rearrangement of a perfect ordered collection is not a perfect ordered collection. We therefore can call an unordered collection perfect if there exists an ordering of that collection that is perfect.

Let $\mathcal{L}_V^* \subset \mathcal{L}_V$ denote the set of perfect collections for V . We will show for V in \mathcal{F}_{2i}

$$(11) \quad |\mathcal{L}_V^*| \geq (2i)! 2^{-i-1} n^{i(i+3)} (1 - 2^i n^{-1}).$$

For a single point Inequality 11 holds. For a plane, P , a pair of points is perfect if they are not collinear. Hence

$$|\mathcal{L}_P^*| = \binom{n^2}{2} - 2n \binom{n}{2} \geq \frac{n^4}{2} (1 - 2n^{-1})$$

and Inequality 11 is true. We continue inductively and assume for $i \geq 2$ any subtorus $W \in \mathcal{F}_{2i-2}$,

$$|\mathcal{L}_W^*| \geq (2i-2)! 2^{-i} n^{(i-1)(i+2)} (1 - 2^{i-1} n^{-1}).$$

For a fixed $W \subset V$ with $W \in \mathcal{F}_{2i-2}$ and a fixed $S' \in \mathcal{L}_W^*$ there are at least $(n-i-1)^{2i}$ possible $v \in V$ such that $S' \cup \{v\}$ is in \mathcal{L}_V^* . For $V \in \mathcal{F}_{2i}$, there are exactly $\binom{2i}{2} n^2$ $W \subset V$ with $W \in \mathcal{F}_{2i-2}$. Then

$$\begin{aligned} |\mathcal{L}_V^*| &= \sum_{W \subset V, W \in \mathcal{F}_{2i-2}} \sum_{S' \in \mathcal{L}_W^*} \sum_{v \in V} \mathbf{1}_{S' \cup \{v\} \text{ is perfect in } V} \\ &\geq \sum_{W \subset V, W \in \mathcal{F}_{2i-2}} \sum_{S' \in \mathcal{L}_W^*} (n-i-1)^{2i} \\ &\geq \sum_{W \subset V, W \in \mathcal{F}_{2i-2}} (2i-2)! 2^{-i} n^{(i-1)(i+2)} (1 - 2^{i-1} n^{-1}) n^{2i} (1 - (i-1)n^{-1}) \\ &\geq \binom{2i}{2} n^2 (2i-2)! 2^{-i} n^{(i-1)(i+2)} (1 - 2^i n^{-1}) \\ &= (2i)! 2^{-i-1} n^{i(i+3)} (1 - 2^i n^{-1}). \end{aligned}$$

Combining Inequalities 10 and 11 gives

$$\begin{aligned}
M_{2i} &\geq |\mathcal{L}_V| p^{i+1} (1 - n^{-\beta_d}) \\
&\geq |\mathcal{L}_V^*| p^{i+1} (1 - n^{-\beta_d}) \\
&\geq (2i)! 2^{-i-1} n^{i(i+3)} p^{i+1} (1 - 2^i n^{-1}) (1 - n^{-\beta_d}). \\
&\geq (2i)! 2^{-i-1} n^{i(i+3)} p^{i+1} (1 - n^{-c_d})
\end{aligned}$$

completing the proof. ■

Proposition 3.2. *Fix $d > 2$ and $j \leq J$. Let $f(n) < \log n$ also satisfy $\lim_{n \rightarrow \infty} f(n) = \infty$. If $p = f(n)n^{-d/(j+1)-j}$, then*

$$\mathbb{P}_p(\mathcal{I}_{2j}) \rightarrow 1.$$

Proof. First we define a sufficient event $E_{2j} \subset \mathcal{I}_{2j}$. If we can show $\mathbb{P}_p(E_{2j}) \rightarrow 1$ for some value of p then we can conclude $\mathbb{P}_p(\mathcal{I}_{2j}) \rightarrow 1$ as well.

For a fixed set of constants $\alpha = \{\alpha_{2j+1}, \dots, \alpha_d\}$, let $V(\alpha)$ denote the subtorus given by

$$V(\alpha) = \{v \in [n]^d : v_i = \alpha_i \text{ for } 2j+1 \leq i \leq d\}.$$

There are n^{d-2j} such subtori. For $\alpha' = \{\alpha'_{2j+1}, \dots, \alpha'_d\}$, if $\alpha \neq \alpha'$, $V(\alpha) \cap V(\alpha') = \emptyset$. Each event $\mathcal{I}_{V(\alpha)}$ will depend only on the nodes in $V(\alpha)$ so the events are independent. The events will all have the same probability $\mathbb{P}_p(\mathcal{I}_{V(\alpha)}) = \mathbb{P}_p(\mathcal{I}_{V(\alpha')})$. We now define the sufficient event

$$E_{2j} = \bigcup_{\alpha} \mathcal{I}_{V(\alpha)}.$$

We will show that $\mathbb{P}_p(E_{2j}) \rightarrow 1$ for sufficiently large p that satisfy the conditions of the proposition. Since $E_{2j} \subset \mathcal{I}_{2j}$ this implies $\mathbb{P}_p(\mathcal{I}_{2j}) \rightarrow 1$ as well.

With this definition we have

$$(12) \quad \mathbb{P}(E_{2j}) = 1 - (1 - M_{2j})^{n^{d-2j}} \geq 1 - e^{-n^{d-2j} M_{2j}}.$$

We finish the proof of Proposition 3.2 by proving that $n^{d-2j} M_{2j} \rightarrow \infty$ when the conditions of the proposition are satisfied.

For $f(n) < \log n$ and $j \leq J$, $p = f(n)n^{-d/(j+1)-j}$ satisfies $p < n^{-2J-\epsilon}$ for some $\epsilon > 0$. Then we may apply Lemma 3.1 to show

$$\begin{aligned}
n^{d-2j} M_{2j} &\geq n^{d-2j} (2j)! 2^{-j-1} n^{j(j+3)} p^{j+1} (1 - n^{-c_d}) \\
&\geq f(n)^{j+1} \rightarrow \infty
\end{aligned}$$
■

If \mathcal{I}_{2j} occurs then \mathcal{C}_{2j} also occurs. Proposition 3.2 implies

$$p_c(2j-1, d) \leq p_c(2j, d) \leq p_{\mathcal{I}}(2j, d) < f(n)n^{-d/(j+1)-j}.$$

The caveat that $f(n) < \log n$ is necessary only for the proof of the proposition. Both $\mathbb{P}_p(\mathcal{I}_{2j})$ and $\mathbb{P}_p(\mathcal{C}_{2j})$ are increasing in p , so the proposition will still be true for faster growing $f(n)$ as long as $p \leq 1$.

3.2. Lower Bound for $p_{\mathcal{I}}(i, d)$. In this section we prove the lower bound for the critical exponent of $p_{\mathcal{I}}(2j, d)$.

First let's start with the simplest possibilities for V : a single node, a line, and a plane.

- For a single node u ,

$$\mathbb{P}_p(\mathcal{I}_{\{u\}}) = p.$$

- For a single line L ,

$$\mathbb{P}_p(\mathcal{I}_L) = \mathbb{P}(\text{Bin}(n, p) \geq 2) \leq \binom{n}{2} p^2 = O(n^2 p^2).$$

- For a single plane P ,

$$\mathbb{P}_p(\mathcal{I}_P) \leq \mathbb{P}(\text{Bin}(n^2, p) \geq 2) \leq 2^{-1} n^4 p^2.$$

Note that a plane is more likely to be internally spanned than a line because a line requires at least two collinear points. The following lemma extends these computations.

Lemma 3.3. *Fix d and $j \leq J$ and let $p = f(n)n^{-d/(j+1)-j}$ for some $f(n) \rightarrow 0$. For $1 \leq i \leq j$,*

$$(13) \quad M_{2i+1} \leq O(n^{(i+1)(i+4)-2} p^{i+2}).$$

$$(14) \quad M_{2i} = (1 + O(n^{-1}))(2i)! 2^{-i-1} n^{i(i+3)} p^{i+1}.$$

Proof. (By induction on i)

We assume the lemma holds for all $0 \leq l \leq 2i - 1$ and show by induction that the formulas hold for dimensions $2i$ and $2i + 1$. Note the lemma holds for a point, a line and a plane, so our base case is covered.

First let's assume a subtorus V is internally spanned. By Lemma 2.4, there exists proper subtori $V_1, V_2 \subset V$ both internally spanned by disjoint non-empty subsets S_1 and S_2 such that $V = \langle V_1, V_2 \rangle$. Let D_V denote the set of possible pairs of such subtori of V with $\dim(V_1) \leq \dim(V_2)$. \mathcal{I}_V can be expressed as a union over D_V of events of the form $\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}$, where \circ denotes the events are mutually disjoint. The union bound gives us:

$$\mathbb{P}_p(\mathcal{I}_V) \leq \mathbb{P}_p \left(\bigcup_{D_V} \mathcal{I}_{V_1} \circ \mathcal{I}_{V_2} \right).$$

By the van den Berg-Kesten inequality [10]

$$\mathbb{P}_p(\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}) \leq \mathbb{P}_p(\mathcal{I}_{V_1}) \mathbb{P}_p(\mathcal{I}_{V_2})$$

and

$$\mathbb{P}_p(\mathcal{I}_V) \leq \sum_{D_V} \mathbb{P}_p(\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}) \leq \sum_{D_V} \mathbb{P}_p(\mathcal{I}_{V_1}) \mathbb{P}_p(\mathcal{I}_{V_2}).$$

For $0 \leq t_1 \leq t_2 < \dim(V)$ let $D_V(t_1, t_2)$ denote the subset of D_V where $\dim(V_1) = t_1$ and $\dim(V_2) = t_2$. Since $\langle V_1 \cup V_2 \rangle$ is a subspace it has dimension at most $t_1 + t_2 + 2$. Therefore if $t_1 + t_2 + 2 < \dim(V)$, then $D_V(t_1, t_2)$ is empty. Otherwise $|D_V(t_1, t_2)| = O(n^{2i-t_1} n^{2i-t_2})$. Then we have

$$(15) \quad \sum_{D_V} \mathbb{P}_p(\mathcal{I}_{V_1}) \mathbb{P}_p(\mathcal{I}_{V_2}) = \sum_{0 \leq t_1 \leq t_2} \sum_{D_V(t_1, t_2)} M_{t_1} M_{t_2} = \sum_{0 \leq t_1 \leq t_2} |D_V(t_1, t_2)| M_{t_1} M_{t_2}.$$

If $V \in \mathcal{F}_{2i}$ we will show the probability \mathcal{I}_V occurs is almost entirely determined by the probability there exists a pair $(V_1, V_2) \in D_V(0, 2i - 2)$ such that $\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}$ occurs.

There exists constants C depending only on d such that

$$|D_V(t_1, t_2)| M_{t_1} M_{t_2} = C n^{4i - t_1 - t_2} M_{t_1} M_{t_2}.$$

If $t_1 = 2l + 1$, then by the induction hypothesis $n^{2i - (2l - 1)} M_{2l - 1} = O(n^{-1} n^{2i - 2l} M_{2l})$ so we may assume that t_1 (and t_2) are both even. Let $t_1 = 2i_1$, and $t_2 = 2i_2$, with $i_1 + i_2 + 1 = i + k$, where $k \leq i_1 \leq i_2 < i \geq 0$. By the induction hypothesis we have an upper bound for M_{2i_1} and M_{2i_2} .

Therefore

$$\begin{aligned} |D_V(t_1, t_2)| M_{t_1} M_{t_2} &\leq C n^{4i - 2i_1 - 2i_2} M_{2i_1} M_{2i_2} \\ &= C (1 + O(n^{-1}))^2 n^{4i - 2i_1 - 2i_2} n^{i_1(i_1 + 3) + i_2(i_2 + 3)} p^{i_1 + 1 + i_2 + 1} \\ &\leq C n^{-5i + k - 1 + i_1^2 + i_2^2} p^{i + 1} \\ &\leq C n^{i(i + 3)} p^{i + 1} n^{k(k - 1) - 2i_1 i_2}. \end{aligned}$$

If $i_1 > 0$, then $k(k - 1) - 2i_1 i_2 \leq -2$. Therefore if $i_1 > 0$

$$(16) \quad |D_V(t_1, t_2)| M_{t_1} M_{t_2} = O(n^{-1}) n^{i(i + 3)} p^{i + 1}.$$

If $t_1 = 0$ then $t_2 = 2i - 2$. There are at most $\binom{2i}{2} (n^{2i} n^2)$ pairs in $D_V(0, 2i - 2)$. Therefore

$$(17) \quad |D_V(0, 2i - 2)| M_0 M_{2i - 2} \leq \binom{2i}{2} (n^{2i} n^2) n^{i(i + 3)} p^{i + 1} (1 + O(n^{-1}))$$

Combining Equations (16) and (17)

$$(18) \quad \sum_{D_V} \mathbb{P}_p(\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}) \leq (1 + O(n^{-1})) \binom{2i}{2} (n^{2i} n^2) n^{i(i + 3)} p^{i + 1}$$

gives an upper bound for M_{2i} . This inequality combines with Lemma 3.1 to prove Equation 14 of Lemma 3.3.

A similar argument shows that for $\dim(V) = 2i + 1$ the sum is dominated by the terms from $D_V(0, 2i)$. If $\dim(V) = 2i + 1$, then there are at most $O(n^{2i + 1} n)$ pairs in $D_V(0, 2i)$. The union bound gives

$$|D_V(0, 2i)| M_0 M_{2i} = O(n^{2i} n) M_0 M_{2i} = O\left(n^{(i + 1)(i + 4) - 2} p^{i + 2}\right),$$

proving 13 of Lemma 3.3. ■

Proposition 3.4. *Fix d and $j \leq J$. For any $f(n) \rightarrow 0$, if $p = f(n)n^{-d/(j+1)-j}$, then*

$$\mathbb{P}_p(\mathcal{I}_{2j}) \rightarrow 0.$$

This proposition implies $p_{\mathcal{I}}(2j, d) > f(n)n^{-d/(j+1)-j}$. Unlike Proposition 3.2, we need a little extra care to claim $p_c(2j, d) > f(n)n^{-d/(j+1)-j}$ (see Section 3.3).

Proof. The union bound gives:

$$\mathbb{P}_p(\mathcal{I}_{2j}) \leq \sum_{V \in \mathcal{F}_{2j}} \mathbb{P}_p(\mathcal{I}_V) \leq \binom{d}{2j} n^{d-2j} M_{2j}.$$

By Lemma 3.3, $M_{2j} = O(f(n)^{j+1}n^{2j-d})$ when $p = f(n)n^{-d/(j+1)-j}$. Then $\mathbb{P}_p(\mathcal{I}_{2j}) = O(f(n)^{j+1}) \rightarrow 0$ which implies $p_{\mathcal{I}}(2j, d) > f(n)n^{-d/(j+1)-j}$. ■

3.3. Bounds for $p_c(2j, d)$. In this section we will show $\mathbb{P}_p(\mathcal{C}_{2j} \setminus \mathcal{I}_{2j}) \rightarrow 0$. We will show that if $\mathbb{P}(\mathcal{I}_{2j}) = 0$ then $\mathbb{P}_p(\mathcal{C}_{2j}) = 0$. By Proposition 3.4 we have for fixed d and $j \leq J$ with $f(n) \rightarrow 0$,

$$p_{\mathcal{I}}(2j, d) \geq f(n)n^{-d/(j+1)-j}$$

for large enough n .

If \mathcal{C}_{2j} occurs then there exists some subtorus with dimension greater than or equal to $2j$ that is internally spanned. The next lemma will show that for any dimension $b > 2j$, $\mathbb{P}_p(\mathcal{I}_b) \rightarrow 0$ as $\mathbb{P}_p(\mathcal{I}_{2j}) \rightarrow 0$. This implies that $\mathbb{P}_p(\mathcal{C}_{2j}) \rightarrow 0$ as well.

Lemma 3.5. *Fix d and $j \leq J$, and let $p = an^{-d/(j+1)-j}$. Then*

$$\mathbb{P}_p(\mathcal{C}_{2j} \setminus \mathcal{I}_{2j}) \rightarrow 0.$$

Proof of Lemma 3.5. If \mathcal{C}_{2j} occurs, then there must be some s -dimensional subtorus V such that \mathcal{I}_s occurs and $s \geq 2j$. Let b be the minimal s and suppose $b > 2j$. By Lemma 2.4, if \mathcal{I}_V occurs for some $V \in \mathcal{F}_b$, there exist V_1 and $V_2 \subset V$ with $\dim(V_1) \leq \dim(V_2) < b$ such that $\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}$ occurs and $\langle V_1 \cup V_2 \rangle = V$. Moreover $\dim(V_2) < 2j$ since b is minimal.

Consider the set of t_1, t_2 such that $0 < t_1 \leq t_2 < 2j$. First suppose $t_1 + t_2 + 2 = b$. Any decrease in either t_1 or t_2 will cause $D_V(t_1, t_2)$ to be empty. We will assume for simplicity $t_1 = 2i_1, t_2 = 2i_2$ and $b = 2i = 2j + 2k$ for some $0 < k < i_1 \leq i_2 < j$.

The expression $n^{2b-t_1-t_2} M_{t_1} M_{t_2}$ decreases if t_1 or t_2 increases.

$$\begin{aligned}
\mathbb{P}_p(\mathcal{I}_V \setminus \mathcal{I}_{2J}) &= O \left(\sum_{t_1+t_2} n^{2b-t_1-t_2} M_{t_1} M_{t_2} \right) \\
&= O \left(\sum_{t_1+t_2=b-2} n^{2b-t_1-t_2} M_{t_1} M_{t_2} \right) \\
&= O \left(n^{4i-2i_1-2i_2+i_1^2+i_2^2+3i_1+3i_2} p^{i_1+i_2+2} \right) \\
&= O \left(n^{-2i_1-2i_2+j^2+k^2+2jk+3j+3k+1} n^{-d-j(j+1)} p^k \right) \\
&= O \left(n^{2j+2k-d+1-2i_1i_2+k(k+1)} \right) \\
&= \left(n^{b-d-1} \right)
\end{aligned}$$

since $1 - 2i_1i_2 + k(k+1) \leq -1$.

There are only finitely many choices for b , and only $O(n^{d-b})$ subtori of dimension b . Therefore the probability there exists an internally spanned subtorus of dimension b tends to zero. This is true for all $b > 2j$. ■

Now we can conclude that $p_c(2j, d)$ is also bounded below $f(n)n^{-d/(j+1)-j}$ for any $f(n) \rightarrow 0$.

4. POISSON APPROXIMATION

Odd and even dimensions will play a significant role in the ensuing computations. To help streamline the presentation we introduce the following notation. Let $\sigma_x = x - 2\lfloor x/2 \rfloor$. If $y = \lfloor x/2 \rfloor$ then we may write $x = 2y - \sigma_x$. This simplified notation will help us write our formula for p_{2j}^r when r is even and when r is odd.

The following lemma considers the probability that \mathcal{I}_V occurs conditioned on some subtorus of V being completely open. Let \mathcal{F}_V^r denote the set of subtori in \mathcal{F}_r that completely contained in V . For $U \in \mathcal{F}_V^r$ let $\mathcal{I}_{U \rightarrow V}$ denote the event that \mathcal{I}_V occurs with initial condition U is completely open.

Lemma 4.1. *Fix $d > 2$, and let $j \leq J$ and $p \leq n^{-2j}$. Fix $r < t \leq 2j$ and let $i = \lceil t/2 \rceil$ and $l = \lceil r/2 \rceil$. For $U \in \mathcal{F}_V^r$,*

$$(19) \quad \mathbb{P}_p(\mathcal{I}_{U \rightarrow V}) = O \left(n^{i^2-l^2+E(t,r)} p^{i-l+e(t,r)} \right).$$

where

$$E(t, r) = \begin{cases} i-l, & t=2i, r=2l \\ i-l-1, & t=2i-1, r=2l \\ 2i, & t=2i, r=2l-1 \\ 0, & t=2i-1, r=2l-1 \end{cases} \quad \text{and} \quad e(t, r) = \begin{cases} 0, & t=2i, r=2l \\ 0, & t=2i-1, r=2l \\ 1, & t=2i, r=2l-1 \\ 0, & t=2i-1, r=2l-1 \end{cases}.$$

The proof is rather long and technical and will be delayed until the appendix.

Lemma 4.2. Fix $d > 2$, and let $j \leq J$ and $p \leq an^{-d/(j+1)-2j}$. Fix $r < s, t \leq 2j$ and suppose $V \in \mathcal{F}_t, W \in \mathcal{F}_s, V \cap W \in \mathcal{F}_r$. Let $i = \lceil t/2 \rceil, k = \lceil s/2 \rceil$ and $l = \lceil r/2 \rceil$. Then if $r < \min(t, s)$.

$$\mathbb{P}_p(\mathcal{I}_V \cap \mathcal{I}_W) = O\left(n^{i^2+k^2-l^2+i+k+l-\sigma_r} p^{i+k-l+1}\right).$$

Otherwise if $r = \min(t, s)$ (w.l.o.g. assume $r = t$) then

$$\mathbb{P}_p(\mathcal{I}_V \cap \mathcal{I}_W) \leq \mathbb{P}(\mathcal{I}_V) \mathbb{P}(\mathcal{I}_{V \rightarrow W}) = O\left(n^{k^2+k+2i-\sigma_t} p^{k+1}\right).$$

This proof is also long and technical and will be delayed until the appendix. However we may conclude from Lemma 4.2 that if $t = s = 2j$, then

$$\begin{aligned} \mathbb{P}_p(\mathcal{I}_V \cap \mathcal{I}_W) &= O\left(n^{2j^2-l^2+2j+l-\sigma_r} p^{j+1+j-l}\right) \\ &= O\left(n^{r-d-\epsilon}\right) n^{(j-l)(j+l+1)} p^{j-l} = o(n^{r-d}) \end{aligned}$$

since $d/(j+1) > j + \epsilon$ for some $\epsilon > 0$.

Proof of Theorem 1.2. We use the Chen-Stein method for approximation by a Poisson distribution. We will use the version found in [13] (Theorem 4.7):

Theorem 4.3. Let X_1, \dots, X_m be positively related indicator variables with $\mathbb{P}(X_i = 1) = p_i$, $Y = \sum_{i=1}^n X_i$, and $\lambda = \mathbb{E}[Y] = \sum_i p_i$. For each $i \in [m]$, let $N_i \subset [m]$ where $i \in N_i$ and X_i is independent of $\{X_j : j \notin N_i\}$. If $p_{ij} := \mathbb{E}[X_i X_j]$ and $Z \sim \text{Po}(\lambda)$, then

$$(20) \quad d_{TV}(Y, Z) \leq 2 \sum_{i=1}^m \left(p_i p_j + \sum_{j \in N_i \setminus \{i\}} p_{ij} \right).$$

Each subtorus $V \in \mathcal{F}_{2j}$ has a dependency set $\Gamma_V \subset \mathcal{F}_{2j}$ such that for $W \in \Gamma_V$, $V \cap W$ is non-empty. For fixed $d > 2$ and $j \leq J$, when $p = an^{-d/(j+1)-j}$ then each subspace $V \in \mathcal{F}_{2j}$ is internally spanned with probability $M_{2j} = (2j)! 2^{-j-1} a^{j+1} n^{2j-d} (1 + o(1))$. Although some dependency exists, we will show that the distribution of the number of subtori with dimension $2j$ which are internally spanned approaches a Poisson distribution.

To fit our random variables with that of the theorem, we let $\mathbf{1}_V$ denote the indicator random variable for the event \mathcal{I}_V . For all $V, W \in \mathcal{F}_{2j}$,

$$p_V = p_W = M_{2j} = (2j)! 2^{-j-1} a^{j+1} n^{2j-d} (1 + o(1))$$

and

$$p_{VW} = \mathbb{E}[\mathbf{1}_V \mathbf{1}_W] = \mathbb{P}_p(\mathcal{I}_V \cap \mathcal{I}_W).$$

Let $Y = \sum_{\Gamma} \mathbf{1}_V$. Then

$$\lambda = \mathbb{E}[Y] = (1 + o(1)) \sum_{\mathcal{F}_{2j}} (2j)! 2^{-j-1} a^{j+1} n^{2j-d} = (1 + o(1)) \binom{d}{2j} (2j)! 2^{-j-1} a^{j+1}.$$

Finally we let $Z \sim \text{Po}(\lambda)$, a Poisson random variable with parameter λ .

The indicator random variables $\mathbf{1}_V$ and $\mathbf{1}_W$ are positively related, so plugging everything into Inequality 20 we get

$$(21) \quad d_{TV}(Y, Z) \leq 2 \sum_{V \in \mathcal{F}_{2j}} \left(p_V p_W + \sum_{W \in \Gamma_V \setminus \{V\}} p_{VW} \right).$$

Let $s_V = |\Gamma_V|$. For every $V, W \in \mathcal{F}_{2j}$, $s_V = s_W$ so we denote this size by s_{2j} , allowing us to rewrite part of Inequality 21 as

$$(22) \quad 2 \sum_{V \in \mathcal{F}_{2j}} p_V p_W = 2 \sum_{V \in \mathcal{F}_{2j}} M_{2j}^2 = O(M_{2j}) \rightarrow 0.$$

The quantity p_{VW} depends only on the dimension $\dim(V \cap W) = r$. We break up Γ_V into subsets Γ_V^r where $W \in \Gamma_V^r$ if $\dim(V \cap W) = r$. Then $p_{VW} = p_{2j}^r$ for $W \in \Gamma_V^r$. For each $0 \leq r < 2j$, let $s_{2j}^r = |\Gamma_V^r| = O(n^{2j-r})$. We have

The remaining part of the right-hand side of Inequality 21 can be written as

$$(23) \quad \sum_{V \in \mathcal{F}_{2j}} \sum_{W \in \Gamma_V \setminus \{V\}} p_{VW} = |\mathcal{F}_{2j}| \sum_{r=0}^{2j} s_{2j}^r p_{2j}^r.$$

For each $0 \leq r < 2j$ we have

$$(24) \quad |\mathcal{F}_{2j}| s_{2j}^r p_{2j}^r = O(n^{d-r}) p_{2j}^r = o(1)$$

by Lemma 4.2. Therefore the righthand side of 21 tends to 0, proving the theorem. \blacksquare

5. PROOFS OF THEOREMS

Theorem 1.1 can be viewed as an immediate corollary of Theorem 1.2 and Lemma 3.5. These combine to show $\mathbb{P}_p(\mathcal{C}_{2j} \setminus \mathcal{I}_{2j}) \rightarrow 0$.

Proof of Theorem 1.3. We will use “sprinkling” as in [4] to show that if \mathcal{I}_{2J} occurs \mathcal{I}_t occurs for $t \geq 2J + 2$. If $d < (J+1)(J+2)$ then for some $\epsilon > 0$, $n^{-d/(J+1)-J} \geq n^{-2J-2+\epsilon}$. For a small $\delta > 0$ let $p = (a + \delta)n^{-d/(J+1)-J}$, $p_1 = an^{-d/(J+1)-J}$ and $p_2 = n^{2J-2+\epsilon/2}$. Then $p_1 + p_2 < p$.

Consider two random initial configurations ω_0^1 and ω_0^2 where each node in $[n]^d$ is open with probability p_1 and p_2 respectively. Let $\tilde{\omega}_0$ denote the random configuration given by $\omega_0^1 \cup \omega_0^2$. This configuration is stochastically dominated by the random configuration, ω_0 , where each node is open with probability p .

For any $V \in \mathcal{F}_{2J}$ there will be cn^{2J+2} neighbors exactly distance 2 away from V . Let \mathcal{N}_V denote the event that at least one of the cn^{2J+2} neighbors is open. Since $p_2 = n^{-2J-2+\epsilon/2}$, then for all $V \in \mathcal{F}_{2J}$

$$\mathbb{P}_{p_2}(\mathcal{N}_V(\omega_0^2)) = 1 - o(1).$$

For every $V \in \mathcal{F}_{2J}$,

$$\mathbb{P}_{p_1, p_2}(\mathcal{I}_{2J+2}(\tilde{\omega}_0) | \mathcal{I}_V(\omega_0^1)) \geq \mathbb{P}_{p_2}(\mathcal{N}_V(\omega_0^2) | \mathcal{I}_V(\omega_0^1)) = 1 - o(1).$$

Therefore we may conclude for any $V \in \mathcal{F}_{2J}$.

$$\mathbb{P}_p(\mathcal{I}_{2J+2} | \mathcal{I}_V) = 1 - o(1).$$

For any $t \in \{2J+2, \dots, d\}$ we let $p_1 = an^{-d/(J+1)-J} - n^{-2J-2+\epsilon/2}$ and $p_k = \frac{1}{2^k} n^{-2J-2+\epsilon/2}$ for $k = d-2J$. Then $p_1 + p_2 + \dots + p_k < p$ and for each k and we can show $\mathbb{P}_p(\mathcal{I}_t | \mathcal{I}_{2J+2}) = 1 - o(1)$. ■

Proof of Theorem 1.4. If $(J+1)(J+2) = d \geq 6$ then

$$d/(J+1) + J = d/(J+2) + (J+1) = 2J+2.$$

Let $p = an^{-2J-2}$, and Y_{2J} denote the number of subtori of dimension $2J$ that are internally spanned.

For a fixed set of k subtori $B_k = \{V_1, \dots, V_k\}$ with each $V_i \in \mathcal{F}_{2J}$ define the event $Q(B_k)$ that a subtori in \mathcal{F}_{2J} is internally spanned if and only if it is in B_k . Let \mathcal{W}_k denote the set of all such B_k .

There are at most $k \binom{d-2J}{2} n^{2J+2}$ nodes near subtori in B_k such that, if open, \mathcal{I}_{2J+2} occurs. Let $N(B_k)$ denote the event that one of these neighboring nodes is open and let $\tilde{\mathcal{I}}_{2J+2} = \bigcup_{B_k \in \mathcal{W}_k} N(B_k)$.

The events $Q(B_k)$ are decreasing on the nodes outside of B_k while the events $N(B_k)$ are increasing on the nodes outside of B_k . By the Fortuin-Kasteleyn-Ginibre [10] inequality

$$\mathbb{P}(N(B_k) \cap Q(B_k)) \leq \mathbb{P}(N(B_k) | Q(B_k)) \mathbb{P}(Q(B_k)).$$

Therefore

$$\begin{aligned} \mathbb{P}(\tilde{\mathcal{I}}_{2J+2} \cap \{Y_{2J} = k\}) &\leq \sum_{B_k \in \mathcal{W}_k} \mathbb{P}(N(B_k) \cap Q(B_k)) \\ &\leq \sum_{B_k \in \mathcal{W}_k} \mathbb{P}(N(B_k) | Q(B_k)) \mathbb{P}(Q(B_k)). \end{aligned}$$

The conditional probability is uniform over all $B_k \in \mathcal{W}_k$. Letting $c = \binom{d-2J}{2}$, and $\lambda = \lambda_J$, we have

$$\mathbb{P}_p(N(B_k) | Q(B_k)) \leq 1 - (1 - an^{-2J-2})^{ckn^{2J+2}} \leq 1 - e^{ack}$$

and

$$\begin{aligned} \mathbb{P}_p(\tilde{\mathcal{I}}_{2J+2} \cap \{Y_{2J} = k\}) &\leq (1 - e^{ack}) \sum_{B_k \in \mathcal{W}_k} \mathbb{P}_p(Q(B_k)) \\ &\leq (1 - e^{ack}) \mathbb{P}_p(Y_{2J} = k) \\ &\leq (1 - e^{-ack}) e^{-\lambda} \lambda^k / k!. \end{aligned}$$

Summing over k gives,

$$\mathbb{P}_p(\tilde{\mathcal{I}}_{2J+2}) \leq e^{-\lambda} \sum_{k=1}^{\infty} (\lambda^k - (\lambda e^{-ac})^k) / k!$$

$$(25) \quad = 1 - e^{\lambda e^{-ac} - \lambda}.$$

By Lemma 2.4 if \mathcal{I}_{2J+2} occurs but $\tilde{\mathcal{I}}_{2J+2}$ does not, then for some $0 < t_1 \leq t_2 \leq 2J+2$ there exists $V \in \mathcal{F}_{2J+2}$ and $V_1, V_2 \subset V$ with $V_1 \in \mathcal{F}_{t_1}$, $V_2 \in \mathcal{F}_{t_2}$ such that $\langle \{V_1, V_2\} \rangle = V$ and $\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}$ occurs.

There are $O(n^{d-2J-2})$ possible V with $O(n^{4J+4-t_1-t_2})$ possible choices for pairs V_1 and V_2 . Then

$$(26) \quad \mathbb{P}_p(\mathcal{I}_{2J+2} \setminus \tilde{\mathcal{I}}_{2J+2}) \leq \sum_{0 < t_1 \leq t_2 \leq 2J+2} O(n^{d+2J+2-t_1-t_2}) M_{t_1} M_{t_2}.$$

First note that similar arguments will give $\mathbb{P}_p(\mathcal{I}_{2J+1}) = O(n^{-1})$. If t_1 and t_2 are both less than $2J+1$ then we may use Lemma 3.3 to show each of the summands is $O(n^{-1})$. Then

$$\mathbb{P}_p(\mathcal{I}_{2J+2} \setminus \tilde{\mathcal{I}}_{2J+2}) \leq \mathbb{P}_p(\mathcal{I}_{2J+1}) + \sum_{t_1, t_2} O(n^{-1}) = O(n^{-1})$$

and we may conclude

$$\mathbb{P}_p(\mathcal{I}_{2J+2}) \leq 1 - e^{\lambda e^{-ac} - \lambda} + O(n^{-1}).$$

With this upper bound for $\mathbb{P}_p(\mathcal{I}_{2J+2})$,

$$\mathbb{P}_p(\mathcal{I}_{2J} \setminus \mathcal{I}_{2J+2}) > 1 - e^{-\lambda_J} - (1 - e^{\lambda e^{-ac} - \lambda_J}) + O(n^{-1}) > 0$$

for large enough n , finishing the proof of part 4 of the Theorem.

For part 5 of the Theorem we will use sprinkling again. Let $p_1 = p_2 = p/2$. There are at least $k(cn^{2J+2} - O(n^{2J+1}))$ nodes in the neighborhood of B_k such that if open and $Q(B_k)$ occurs, then \mathcal{I}_{2J+2} also occurs. Letting $\gamma = \lambda(J, d, a/2)$, we have

$$\begin{aligned} \mathbb{P}_p(\mathcal{I}_{2J+2} \cap \{Y_{2J+2} = k\}) &\geq \mathbb{P}_{p/2}(Y_{2J+2} = k) \left(1 - (1 - p/2)^{ckn^{2J+2} - O(n^{2J+1})}\right) \\ &\geq (1 - e^{ack/5}) e^{-\gamma} \gamma^k / k!. \end{aligned}$$

Summing over k gives

$$(27) \quad \mathbb{P}_p(\mathcal{I}_{2J+2}) \geq 1 - e^{\gamma e^{-ac/5} - \gamma} > 0.$$

Lastly for part 6 we use sprinkling again. The proof follows precisely from that of Theorem 1.3. ■

Proof of Theorem 1.5. We will provide a sketch of the proof of this theorem as it follows similar arguments found in prior proofs.

Let $J \geq 2$. Consider $V \in \mathcal{F}_{2J+2k}$. Let N_{2J+2k} denote the probability that $\mathcal{I}_V \setminus \mathcal{I}_{2J+2}$ occurs. In particular $N_{2J+2} = 0$.

For $k = 2$, if both a $2J$ - and 2 - dimensional subtori is spanned within V then it is possible for \mathcal{I}_V to occur. Lower dimensional combinations are possible but less likely. In particular

$$\begin{aligned} N_{2J+4} &= O(n^4 N_{2J} n^{2J+2} N_2) \\ &= O(n^{6-d}). \end{aligned}$$

For $2 < k < d - 2J$, there are $O(n^2)$ subtori of dimension $2J + 2k - 2$ such that if open, the $2J + 2k$ dimensional subtori will become open with high probability. Again, lower dimensional combinations are less likely. Therefore,

$$N_{2J+2k} = O(n^2 n^{2k-d}).$$

For each k , $|\mathcal{F}_{2J+2k}| = O(n^{d-2J-2k})$. The probability that at least one of the $V \in \mathcal{F}_{2J+2k}$ is internally spanned without \mathcal{I}_{2J+2} occurring is at most

$$\mathbb{P}_p(\mathcal{I}_d \setminus \mathcal{I}_{2J+2}) = O(n^{2-2J}).$$

This tends to 0 if $J > 1$.

If $J = 1$ and $d = 6$, then two planes can be exactly distance 2 apart and span all of n^d without any subtori in \mathcal{F}_4 being internally spanned. At least $1/10$ of all pairs of planes are exactly distance 2 apart.

For some constant $\lambda > 0$

$$\mathbb{P}_p(\mathcal{I}_6 \cap \{Y_2 = 2\}) \geq 1/15 \mathbb{P}_p(Y_2 = 2) > \frac{1}{30} e^{-\lambda} \lambda^2.$$

The probability that a 2-neighbor of the two internally spanned planes is open decreases if we condition on those two planes being the only internally spanned planes. Therefore

$$\mathbb{P}_p(\mathcal{I}_6 \setminus \mathcal{I}_4) \geq \mathbb{P}_p(Y_2 = 2 \cap \mathcal{I}_6) (1 - an^{-4})^{36n^4} \geq \frac{1}{60} e^{-\lambda} \lambda^2 e^{-36a} > 0.$$

The constants chosen were not optimized. ■

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7. APPENDIX

In this appendix we provide proofs for Lemmas 4.1 and 4.2

Proof of Lemma 4.1 (by induction on $t = \dim(V)$). In order to understand how $\mathcal{I}_{U \rightarrow V}$ occurs we recall Lemma 2.4. Either there is a subtori, V_1 of V that is not contained in U such that $\langle V_1, U \rangle = V$ and $\mathcal{I}_{U \rightarrow V_1}$ occurs; or there are two subtori of V , neither contained in U , such that $V = \langle \{V_1, V_2\} \rangle$ and $\mathcal{I}_{U \rightarrow V_1} \circ \mathcal{I}_{U \rightarrow V_2}$ occurs. In both these cases we must consider whether the subtori intersect U .

Consider the following five sets:

- $D^1 = D^1(t_1) := \{V_1 \in \mathcal{F}_V^{t_1} : \langle U, V_1 \rangle = V, U \cap V_1 = \emptyset\},$
- $D^2 = D^2(t_1, r_1) := \{V_1 \in \mathcal{F}_V^{t_1} : \langle U, V_1 \rangle = V, \dim(U \cap V_1) = r_1\},$
- $D^3 = D^3(t_1, t_2) := \{V_1 \in \mathcal{F}_V^{t_1}, V_2 \in \mathcal{F}_V^{t_2} : \langle V_1, V_2 \rangle = V, U \cap V_1 = U \cap V_2 = \emptyset\},$

- $D^4 = D^4(t_1, t_2, r_1) := \{V_1 \in \mathcal{F}_V^{t_1}, V_2 \in \mathcal{F}_V^{t_2} : \langle V_1, V_2 \rangle = V, \dim(U \cap V_1) = r_1, U \cap V_2 = \emptyset\},$
- $D^5 = D^5(t_1, t_2, r_1, r_2) := \{V_1 \in \mathcal{F}_V^{t_1}, V_2 \in \mathcal{F}_V^{t_2} : \langle V_1, V_2 \rangle = V, \dim(U \cap V_1) = r_1, \dim(U \cap V_2) = r_2\}.$

If $\mathcal{I}_{U \rightarrow V}$ occurs, then either for some t_1, t_2, r_1, r_2 one of the following must occur:

- \mathcal{I}_{V_1} occurs for some $V_1 \in D^1,$
- $\mathcal{I}_{U \rightarrow V_2}$ occurs for some $V_1 \in D^2,$
- $\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}$ occurs for some $(V_1, V_2) \in D^3,$
- $\mathcal{I}_{U \rightarrow V_1} \circ \mathcal{I}_{V_2}$ occurs for some $(V_1, V_2) \in D^4,$
- or $\mathcal{I}_{U \rightarrow V_2} \circ \mathcal{I}_{U \rightarrow V_2}$ occurs for some $(V_1, V_2) \in D^5.$

For each of the sets $D^1, D^2, D^3, D^4,$ and D^5 we consider the least upper bound of the probability of these associated events $P^1, P^2, P^3, P^4,$ and P^5 . Then

$$(28) \quad \mathbb{P}_p(\mathcal{I}_{U \rightarrow V}) \leq \sum_{t_1, t_2, r_1, r_2} |D^1|P^1 + |D^2|P^2 + |D^3|P^3 + |D^4|P^4 + |D^5|P^5.$$

For fixed t_1, t_2, r_1, r_2 the sizes of each of these sets is

- $|D^1| = O(n^{t-t_1}),$
- $|D^2| = O(n^{r-r_1}),$
- $|D^3| = O(n^{2t-t_1-t_2}),$
- $|D^4| = O(n^{r-r_1+t-t_2}),$
- $|D^5| = O(n^{2r-r_1-r_2}).$

Let $i_1 = \lceil t_1/2 \rceil, i_2 = \lceil t_2/2 \rceil, l_1 = \lceil r_1/2 \rceil,$ and $l_2 = \lceil r_2/2 \rceil$. Then

$$(29) \quad |D^1|P^1 = O\left(n^{2i-\sigma_t-2i_1+i_1^2+3i_1}p^{i_1+1}\right),$$

$$(30) \quad |D^2|P^2 = O\left(n^{2l-\sigma_r-2l_1+\sigma_{r_1}+i_1^2-l_1^2+E(t_1, r_1)}p^{i_1-l_1+e(t_1, r_1)}\right),$$

$$(31) \quad |D^3|P^3 = O\left(n^{4i-2\sigma_t+i_1^2+i_2^2+i_1+i_2}p^{i_1+i_2+2}\right),$$

$$(32) \quad |D^4|P^4 = O\left(n^{2i-\sigma_t+2l-\sigma_r-2l_2+\sigma_{r_2}+i_1^2+i_1+i_2^2-l_2^2+E(t_2, r_2)}p^{i_1+i_2-l_2+1+e(t_2, r_2)}\right),$$

$$(33) \quad |D^5|P^5 = O\left(n^{4l-2\sigma_r-2l_1+\sigma_{r_1}-2l_2+\sigma_{r_2}+i_1^2+i_2^2-l_1^2-l_2^2+E(t_1, r_1)+E(t_2, r_2)}p^{i_1+i_2-l_1-l_2+e(t_1, r_1)+e(t_2, r_2)}\right).$$

Letting $Q(t, r) = n^{i^2-l^2+E(t, r)}p^{i-l+e(t, r)},$ we show for Equations 29 through 33, the right hand side of each of these expressions is $O(Q(t, r))$. Using the lower bound $p < n^{-2j-\epsilon}$ we see that each of these expressions decreases if i_1 (or i_2 where defined) increases as long as t_1 (and t_2) is less than t . It suffices to show these equations are $O(Q(t, r))$ for minimal i_1 (and i_2).

On the other hand if we decrease l_1 or l_2 there must be a coupled decrease in i_1 or i_2 . Equations 30, 32, and 33 each decrease with the coupled decrease of l_1 and i_1 or l_2

t	t_1	t_2	$\min(i_1 + i_2)$
$2i$	$2i_1$	$2i_2$	$i - 1$
$2i$	$2i_1$	$2i_2 - 1$	i
$2i$	$2i_1 - 1$	$2i_2$	i
$2i$	$2i_1 - 1$	$2i_2 - 1$	i
$2i - 1$	$2i_1$	$2i_2$	$i - 1$
$2i - 1$	$2i_1$	$2i_2 - 1$	$i - 1$
$2i - 1$	$2i_1 - 1$	$2i_2$	$i - 1$
$2i - 1$	$2i_1 - 1$	$2i_2 - 1$	i

TABLE 1. Minimal pairs of i_1 and i_2 .

and i_2). For each of these cases we show each of the relevant equations are $O(Q(t, r))$ for maximal l_1 (or maximal l_2).

Suppose $\langle V_1, V_2 \rangle = V$ with $\dim(V_s) = 2i_s - \sigma_{t_s}$ for $s = 1, 2$. Then

$$2i_1 - \sigma_{t_1} + 2i_2 - \sigma_{t_2} \geq 2i - \sigma_t - 2.$$

The minimal possible values for i_1 and i_2 are given in the table in Fig. 1.

Equation 29 $= O(Q(t, r))$.. For $V_1 \in D^1$, $\langle V_1, U \rangle = V$ only if $r + t_1 \geq t - 2$. The minimal possible i_1 given r and t will depend on the parity of t_1, r , and t . The dependence follows from Fig. 1

For Equation 29 the right hand side is equal to

$$(34) \quad O\left(Q(t, r)n^{2i-\sigma_t+i_1^2+i_1-i^2+l^2-E(t,r)}p^{i_1-i+l-e(t,r)+1}\right).$$

If $t = 2i$ and $r = 2l - 1$ then $e(t, r) = 1$ and $i_1 \geq i - l$. The minimal $i_1 = i - l$ plugged into 34 gives

$$O\left(Q(t, r)n^{i_1-2i_1l}\right) = O(Q(t, r))$$

since $l > 0$. Otherwise $e(t, r) = 0$ and we may assume $i_1 = i - l - 1$. Plugging into 34 gives

$$O\left(Q(t, r)n^{-2i_1l-2i+1-E(t,r)}\right) = O(Q(t, r)).$$

Equation 30 $= O(Q(t, r))$.. For Equation 30 the right hand side is equal to

$$(35) \quad O(Q(t, r)R_2(t, r, t_1, r_1)).$$

where

$$R_2(t, r, t_1, r_1) = n^{2l-\sigma_r-2l_1+\sigma_{r_1}+i_1^2-l_1^2+E(t_1, r_1)-i^2+l^2-E(t, r)}p^{i_1-l_1-i+l+e(t_1, r_1)-e(t, r)}.$$

We show $R_2(t, r, t_1, r_2) = O(1)$.

For $V_1 \in D_{U \rightarrow V}^1(t_1, r_1)$ we have the equality $t_1 - r_1 = t - r$. The parity of the difference must match. For each of the possibilities listed we have the corresponding $R_2(t, r, t_1, r_1)$ each of which are $O(1)$.

t	r	t_1	r_1	$R_2(t, r, t_1, r_1)$
$2i$	$2l$	$2i_1$	$2l_1$	$n^{2l-2l_1+i_1^2-l_1^2-i^2+l^2+i_1-l_1-i+l}$
$2i$	$2l$	$2i_1 - 1$	$2l_1 - 1$	$n^{2l-2l_1+i_1^2-l_1^2-i^2+l^2-i+l+1}$
$2i - 1$	$2l$	$2i_1 - 1$	$2l_1$	$n^{2l-2l_1+i_1^2-l_1^2-i^2+l^2}$
$2i - 1$	$2l$	$2i_1$	$2l_1 - 1$	$n^{2l-2l_1+i_1^2-l_1^2-i^2+l^2+2i_1+2}$
$2i$	$2l - 1$	$2i_1 - 1$	$2l_1$	$n^{2l-2l_1+i_1^2-l_1^2-i^2+l^2-2-2i}$
$2i$	$2l - 1$	$2i_1$	$2l_1 - 1$	$n^{2l-2l_1+i_1^2-l_1^2-i^2+l^2+2i_1-2i}$
$2i - 1$	$2l - 1$	$2i_1$	$2l_1$	$n^{2l-2l_1+i_1^2-l_1^2-i^2+l^2-1}$
$2i - 1$	$2l - 1$	$2i_1 - 1$	$2l_1 - 1$	$n^{2l-2l_1+i_1^2-l_1^2-i^2+l^2}$

TABLE 2. $t_1 - r_1 = t - r$

Equation 31 $= O(Q(t, r))$. The right hand side of Equation 31 with $Q(t, r)$ factored out is

$$O(Q(t, r)R_3(t, r, t_1, t_2))$$

where

$$(36) \quad R_3(t, r, t_1, t_2) = n^{i_1^2+i_2^2-i^2+l^2-E(t,r)+i_1+i_2+4i-2\sigma_t} p^{i_1+i_2+2-i+l-e(t,r)}.$$

In all cases $i_1 + i_2 \geq i - 1$. If $e(t, r) \neq 1$ and $i_1 + i_2 = i - 1$, then

$$R_3(t, r, t_1, t_2) \leq n^{l^2-2i_1i_2+2i} p^{l+1} = O(1).$$

Otherwise if $e(t, r) = 1$ then $E(t, r) = 2i$ and we have

$$R_3(t, r, t_1, t_2) \leq n^{l^2-2i_1i_2} p^l = O(1).$$

Equation 32 $= O(Q(t, r))$. The right hand side of Equation 32 with $Q(t, r)$ factored out is

$$O(Q(t, r)R_4(t, r, t_1, t_2, r_2))$$

where

$$(37) \quad R_4(t, r, t_1, t_2, r_2) = n^{2i-\sigma_t+2l-\sigma_r-2l_2+\sigma_{r_2}+i_1^2+i_1+i_2^2-l_2^2-(i^2-l^2)+E(t_2, r_2)-E(t, r)} p^{i_1+i_2-l_2+1-i+l-e(t, r)+e(t_2, r_2)}.$$

We simplify Equation 37 when either $i_1 + i_2 = i - 1$ (see Table 3) or $i_1 + i_2 = i$ (see Table 4) using

$$R_4(t, r, t_1, t_2, r_2) = n^{-\sigma_t-\sigma_r+\sigma_{r_2}+i_1-2i_1i_2+1+E(t_2, r_2)-E(t, r)} p^{e(t_2, r_2)-e(t, r)} n^{(l-l_2)(l+l_2+2)} p^{l-l_2}$$

when $i_1 + i_2 = i - 1$ and

$$R_4(t, r, t_1, t_2, r_2) = n^{-\sigma_t-\sigma_r+\sigma_{r_2}+i_1-2i_1i_2+E(t_2, r_2)-E(t, r)} n^{2i} p^{1+e(t_2, r_2)-e(t, r)} n^{(l-l_2)(l+l_2+2)} p^{l-l_2}$$

when $i_1 + i_2 = i$.

t	r	t_1	t_2	r_2	$R_4(t, r, t_1, t_2, r_2)$
$2i$	$2l$	$2i_1$	$2i_2$	$2l_2$	$n^{-2i_1i_2}n^{(l-l_2)(l+l_2+3)}p^{l-l_2}$
$2i$	$2l$	$2i_1$	$2i_2$	$2l_2 - 1$	$n^{l+i_2+1-2i_1i_2}pn^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i$	$2l - 1$	$2i_1$	$2i_2$	$2l_2$	$n^{-2i_1i_2+1+l-i}n^{(l-l_2-1)(l+l_2+2)}p^{l-l_2-1}$
$2i$	$2l - 1$	$2i_1$	$2i_2$	$2l_2 - 1$	$n^{-2i_1i_2+i_2-i}n^{(l-l_2)l+2l_2+2}p^{l-l_2}$
$2i - 1$	$2l$	$2i_1$	$2i_2$	$2l_2$	$n^{-2i_1i_2}n^{(l-l_2)(l+l_2+3)}p^{l-l_2}$
$2i - 1$	$2l$	$2i_1$	$2i_2$	$2l_2 - 1$	$n^{-2i_1i_2}n^{i_2+l+1}pn^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i - 1$	$2l - 1$	$2i_1$	$2i_2$	$2l_2$	$n^{i_1+i_2-2i_1i_2-1-l_2}n^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i - 1$	$2l - 1$	$2i_1$	$2i_2$	$2l_2 - 1$	$n^{i_1-2i_1i_2+2i_2}pn^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i - 1$	$2l$	$2i_1 - 1$	$2i_2$	$2l_2$	$n^{i_1(1-i_2)+i_2(1-i_1)-1-l_2+l-i}n^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i - 1$	$2l$	$2i_1 - 1$	$2i_2$	$2l_2 - 1$	$n^{-2i_1i_2}n^{1+i_2+l}pn^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i - 1$	$2l - 1$	$2i_1 - 1$	$2i_2$	$2l_2$	$n^{i_1+i_2-3-2i_1i_2-l}n^{(l-l_2)(l+l_2+3)}p^{l-l_2}$
$2i - 1$	$2l - 1$	$2i_1 - 1$	$2i_2$	$2l_2 - 1$	$n^{i_1-2i_1i_2}n^{2i_2}pn^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i - 1$	$2l$	$2i_1$	$2i_2 - 1$	$2l_2$	$n^{-2i_1i_2-1}n^{(l-l_2)(l+l_2+3)}p^{l-l_2}$
$2i - 1$	$2l$	$2i_1$	$2i_2 - 1$	$2l_2 - 1$	$n^{(i_1-2i_1i_2)-(i-l_2-2)}n^{(l-l_2)(l+l_2+3)}p^{l-l_2}$
$2i - 1$	$2l - 1$	$2i_1$	$2i_2 - 1$	$2l_2$	$n^{i_1+i_2-2i_1i_2-l_2-2}n^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i - 1$	$2l - 1$	$2i_1$	$2i_2 - 1$	$2l_2 - 1$	$n^{i_1-2i_1i_2}n^{(l-l_2)(l+l_2+2)}p^{l-l_2}$

TABLE 3. $i_1 + i_2 = i - 1$.

t	r	t_1	t_2	r_2	$R_4(t, r, t_1, t_2, r_2)$
$2i$	$2l$	$2i_1 - 1$	$2i_2$	$2l_2$	$n^{-2i_1i_2}n^{2i}pn^{(l-l_2)(l+l_2+3)}p^{l-l_2}$
$2i$	$2l$	$2i_1 - 1$	$2i_2$	$2l_2 - 1$	$n^{i_2-l+1-2i_1i_2}n^{2i}p^2n^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i$	$2l - 1$	$2i_1 - 1$	$2i_2$	$2l_2$	$n^{-2i_1i_2-l_2-i-1}n^{2i}pn^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i$	$2l - 1$	$2i_1 - 1$	$2i_2$	$2l_2 - 1$	$n^{-2i_1i_2+i_2-i}n^{2i}pn^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i$	$2l$	$2i_1$	$2i_2 - 1$	$2l_2$	$n^{-2i_1i_2-1}n^{2i}pn^{(l-l_2)(l+l_2+3)}p^{l-l_2}$
$2i$	$2l$	$2i_1$	$2i_2 - 1$	$2l_2 - 1$	$n^{i_1-2i_1i_2+l+1-i}n^{2i}pn^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i$	$2l - 1$	$2i_1$	$2i_2 - 1$	$2l_2$	$n^{i_1+i_2-2i_1i_2-l_2-2}n^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i$	$2l - 1$	$2i_1$	$2i_2 - 1$	$2l_2 - 1$	$n^{i_1-2i_1i_2}n^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i$	$2l$	$2i_1 - 1$	$2i_2 - 1$	$2l_2$	$n^{-2i_1i_2-1}n^{2i}pn^{(l-l_2)(l+l_2+3)}p^{l-l_2}$
$2i$	$2l$	$2i_1 - 1$	$2i_2 - 1$	$2l_2 - 1$	$n^{-2i_1i_2-i_2+l_2+1}n^{2i}pn^{(l-l_2)(l+l_2+3)}p^{l-l_2}$
$2i$	$2l - 1$	$2i_1 - 1$	$2i_2 - 1$	$2l_2$	$n^{i_1+i_2-2i_1i_2-l_2-1}n^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i$	$2l - 1$	$2i_1 - 1$	$2i_2 - 1$	$2l_2 - 1$	$n^{i_1-2i_1i_2}n^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i - 1$	$2l$	$2i_1 - 1$	$2i_2 - 1$	$2l_2$	$n^{-2i_1i_2-1}n^{2i}pn^{(l-l_2)(l+l_2+3)}p^{l-l_2}$
$2i - 1$	$2l$	$2i_1 - 1$	$2i_2 - 1$	$2l_2 - 1$	$n^{i_1-2i_1i_2+l-i+1}n^{2i}pn^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i - 1$	$2l - 1$	$2i_1 - 1$	$2i_2 - 1$	$2l_2$	$n^{i_1+i_2-2i_1i_2-l_2-3}n^{2i}pn^{(l-l_2)(l+l_2+2)}p^{l-l_2}$
$2i - 1$	$2l - 1$	$2i_1 - 1$	$2i_2 - 1$	$2l_2 - 1$	$n^{i_1-2i_1i_2-1}n^{2i}pn^{(l-l_2)(l+l_2+2)}p^{l-l_2}$

TABLE 4. $i_1 + i_2 = i$.

Equation 33 $= O(Q(t, r))$. The last case to check is when both V_1 and V_2 intersect U and neither V_1 nor V_2 is a subtorus of U .

Factoring $Q(t, r)$ from the right hand side of Equation 33 gives

$$O(Q(t, r)R_5(t, r, t_1, r_1, t_2, r_2))$$

where

$$(38) \quad R_5(t, r, t_1, r_1, t_2, r_2) = n^{2r-r_1-r_2+i_1^2+i_2^2-i^2-l_1^2-l_2^2+l^2+F(t, r, t_1, r_1, t_2, r_2)} p^{i_1+i_2-i-l_1-l_2+l+f(t, t_1, t_2, r, r_1, r_2)}$$

and

$$\begin{aligned} F(t, r, t_1, r_1, t_2, r_2) &= E(t_1, r_1) + E(t_2, r_2) - E(t, r), \\ f(t, t_1, t_2, r, r_1, r_2) &= e(t_1, r_1) + e(t_2, r_2) - e(t, r). \end{aligned}$$

For a subtorus V , let $I(V)$ denote the set of indices of V such that if $v, w \in V$ then $v_i = w_i$ for $i \in I(V)$. We call these the fixed indices of V . The indices in $[d] \setminus I(V)$ are the free indices and the size is equal to the dimension of V .

Define $s_1 = t_1 - r_1$ and $s_2 = t_2 - r_2$. Both s_1 and s_2 are strictly positive. These values correspond to the number of free indices not shared with U of V_1 and V_2 respectively. Let $s_1 = 2k_1 - \sigma_{s_1}$ and $s_2 = 2k_2 - \sigma_{s_2}$. Then

$$t_1 = 2k_1 - \sigma_{s_1} + 2l_1 - \sigma_{r_1}$$

and

$$t_2 = 2k_2 - \sigma_{s_2} + 2l_2 - \sigma_{r_2}.$$

Equation 38 becomes:

$$(39) \quad n^{2r-r_1-r_2+k_1^2+k_2^2+2k_1l_1+2k_2l_2+l^2-i^2+\tilde{F}(t, r, s_1, r_1, s_2, r_2)} p^{k_1+k_2-i+l-e(t, r)}$$

where

$$\begin{aligned} \tilde{F}(t, r, s_1, r_1, s_2, r_2) &= \\ F(t, r, s_1 + r_1, r_1, s_2 + r_2, r_2) &+ \sigma_{s_1}\sigma_{r_1}(1 - 2l_1 - 2k_1) + \sigma_{s_2}\sigma_{r_2}(1 - 2l_2 - 2k_2). \end{aligned}$$

The benefit of this form is that it immediately follows that Equation 39 decreases if k_1 or k_2 increases or if l_1 or l_2 decreases. It suffices to show that Equation 39 is $O(1)$ for all values (s_1, s_2, r_1, r_2) with

$$s_1 + s_2 = t - r \text{ or } s_1 + s_2 = t - r + 1,$$

$$r_1 = r \text{ or } r_1 = r - 1,$$

$$r_2 = r \text{ or } r_2 = r - 1.$$

From these basic configurations we may either increase s_1 and/or s_2 by a multiple of 2 and/or decrease r_1 and/or r_2 by a multiple of 2 to reach any other configuration. All of these possibilities are listed in Tables 5 through 10. Each can be seen to be $O(1)$. ■

t	r	s_1	r_1	s_2	r_2	$R_5(t, r, s_1 + r_1, r_1, s_2 + r_2, r_2)$
$2i$	$2l$	$2k_1$	$2l$	$2k_2$	$2l$	$n^{-2k_1k_2}$
$2i$	$2l$	$2k_1$	$2l - 1$	$2k_2$	$2l$	$n^{-2k_1k_2+1-k_1}$
$2i$	$2l$	$2k_1$	$2l - 1$	$2k_2$	$2l - 1$	$n^{-2k_1k_2+2-k_1-k_2}$
$2i$	$2l$	$2k_1 - 1$	$2l$	$2k_2 - 1$	$2l$	$n^{-2k_1k_2+2i}p$
$2i$	$2l$	$2k_1 - 1$	$2l - 1$	$2k_2 - 1$	$2l$	$n^{-2k_1k_2+1-k_1+2i}p$
$2i$	$2l$	$2k_1 - 1$	$2l - 1$	$2k_2 - 1$	$2l - 1$	$n^{-2k_1k_2+2-k_1-k_2+2i}p$

TABLE 5. $s_1 + s_2 = t - r$, $t = 2i, r = 2l$

t	r	s_1	r_1	s_2	r_2	$R_5(t, r, s_1 + r_1, r_1, s_2 + r_2, r_2)$
$2i - 1$	$2l$	$2k_1 - 1$	$2l$	$2k_2$	$2l$	$n^{-2k_1k_2}$
$2i - 1$	$2l$	$2k_1 - 1$	$2l - 1$	$2k_2$	$2l$	$n^{-2k_1k_2+1-k_1}$
$2i - 1$	$2l$	$2k_1 - 1$	$2l$	$2k_2$	$2l - 1$	$n^{-2k_1k_2+1-k_2}$
$2i - 1$	$2l$	$2k_1 - 1$	$2l - 1$	$2k_2$	$2l - 1$	$n^{-2k_1k_2+2-k_1-k_2}$

TABLE 6. $s_1 + s_2 = t - r$, $t = 2i - 1, r = 2l$

t	r	s_1	r_1	s_2	r_2	$R_5(t, r, s_1 + r_1, r_1, s_2 + r_2, r_2)$
$2i$	$2l - 1$	$2k_1 - 1$	$2l - 1$	$2k_2$	$2l - 1$	$n^{-2k_1k_2}$
$2i$	$2l - 1$	$2k_1 - 1$	$2l - 2$	$2k_2$	$2l - 1$	$n^{-2k_1k_2+1-k_1}$
$2i$	$2l - 1$	$2k_1 - 1$	$2l - 1$	$2k_2$	$2l - 2$	$n^{-2k_1k_2+1-k_2}$
$2i$	$2l - 1$	$2k_1 - 1$	$2l - 2$	$2k_2$	$2l - 2$	$n^{-2k_1k_2+2-k_1-k_2}$

TABLE 7. $s_1 + s_2 = t - r$, $t = 2i, r = 2l - 1$

t	r	s_1	r_1	s_2	r_2	$R_5(t, r, s_1 + r_1, r_1, s_2 + r_2, r_2)$
$2i$	$2l$	$2k_1 - 1$	$2l$	$2k_2$	$2l$	$n^{-2k_1k_2+1+2i}p$
$2i$	$2l$	$2k_1 - 1$	$2l - 1$	$2k_2$	$2l$	$n^{-2k_1k_2+2-k_1+2i}p$
$2i$	$2l$	$2k_1 - 1$	$2l$	$2k_2$	$2l - 1$	$n^{-2k_1k_2+2-k_2+2i}p$
$2i$	$2l$	$2k_1 - 1$	$2l - 1$	$2k_2$	$2l - 1$	$n^{-2k_1k_2+3-k_1-k_2+2i}p$

TABLE 8. $s_1 + s_2 = t - r + 1$, $t = 2i, r = 2l$

Proof of Lemma 4.2 (by induction on t). We partition \tilde{D}_V into three disjoint subsets. There are $O(n^{2t-t_1-t_2})$ pairs in $D_{V,W}^1(t_1, t_2)$ where $V_1 \cap W = V_2 \cap W = \emptyset$. There are $O(n^{t-t_1+r-r_2})$ pairs in $D_{V,W}^2(t_1, t_2, r_2)$ where $V_1 \cap W = \emptyset$ and $V_2 \cap W = U_2$ has $\dim(U_2) = r_2$. Finally there are $O(n^{2r-r_1-r_2})$ pairs in $D_{V,W}^3(t_1, t_2, r_1, r_2)$ where $V_1 \cap W = U_1$ and $\dim(U_1) = r_1$ and $V_2 \cap W = U_2$ and $\dim(U_2) = r_2$.

With these partitions we have

t	r	s_1	r_1	s_2	r_2	$R_5(t, r, s_1 + r_1, r_1, s_2 + r_2, r_2)$
$2i - 1$	$2l$	$2k_1$	$2l$	$2k_2$	$2l$	$n^{-2k_1k_2+1}$
$2i - 1$	$2l$	$2k_1$	$2l - 1$	$2k_2$	$2l$	$n^{-2k_1k_2+2-k_2}$
$2i - 1$	$2l$	$2k_1$	$2l - 1$	$2k_2$	$2l - 1$	$n^{-2k_1k_2+3-k_1-k_2}$
$2i - 1$	$2l$	$2k_1 - 1$	$2l$	$2k_2 - 1$	$2l$	$n^{-2k_1k_2+1+2i}p$
$2i - 1$	$2l$	$2k_1 - 1$	$2l - 1$	$2k_2 - 1$	$2l$	$n^{-2k_1k_2+2-k_1+2i}p$
$2i - 1$	$2l$	$2k_1 - 1$	$2l - 1$	$2k_2 - 1$	$2l - 1$	$n^{-2k_1k_2+3-k_1-k_2+2i}p$

TABLE 9. $s_1 + s_2 = t - r + 1$, $t = 2i - 1$, $r = 2l$

t	r	s_1	r_1	s_2	r_2	$R_5(t, r, s_1 + r_1, r_1, s_2 + r_2, r_2)$
$2i$	$2l - 1$	$2k_1$	$2l - 1$	$2k_2$	$2l - 1$	$n^{-2k_1k_2+1}$
$2i$	$2l - 1$	$2k_1$	$2l - 2$	$2k_2$	$2l - 1$	$n^{-2k_1k_2+2-k_1}$
$2i$	$2l - 1$	$2k_1$	$2l - 2$	$2k_2$	$2l - 2$	$n^{-2k_1k_2+3-k_1-k_2}$
$2i$	$2l - 1$	$2k_1 - 1$	$2l - 1$	$2k_2 - 1$	$2l - 1$	$n^{-2k_1k_2+2+2i}p$
$2i$	$2l - 1$	$2k_1 - 1$	$2l - 2$	$2k_2 - 1$	$2l - 1$	$n^{-2k_1k_2+k_1+3+2i}p$
$2i$	$2l - 1$	$2k_1 - 1$	$2l - 2$	$2k_2 - 1$	$2l - 2$	$n^{-2k_1k_2+k_1+k_2+4+2i}p$

TABLE 10. $s_1 + s_2 = t - r + 1$, $t = 2i$, $r = 2l - 1$

$$\begin{aligned}
(40) \quad \mathbb{P}_p(\mathcal{I}_V \cap \mathcal{I}_W) &\leq \sum_{\tilde{D}_V} \mathbb{P}_p(\{\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}\} \cap \mathcal{I}_W) \\
(41) \quad &\leq \sum_{D_{V,W}^1(t_1, t_2)} \mathbb{P}_p(\{\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}\} \cap \mathcal{I}_W) \\
(42) \quad &+ \sum_{D_{V,W}^2(t_1, t_2, r_2)} \mathbb{P}_p(\{\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}\} \cap \mathcal{I}_W) \\
&+ \sum_{D_{V,W}^3(t_1, t_2, r_1, r_2)} \mathbb{P}_p(\{\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}\} \cap \mathcal{I}_W)
\end{aligned}$$

Upper bounds for Line 40 and 41 are relatively straightforward.

As before we define $i = \lceil t/2 \rceil$, $k = \lceil s/2 \rceil$, $l = \lceil r/2 \rceil$ and $i_1 = \lceil t_1/2 \rceil$, $i_2 = \lceil t_2/2 \rceil$, $l_1 = \lceil r_1/2 \rceil$, $l_2 = \lceil r_2/2 \rceil$. Let

$$Q(t, s, r) = n^{i^2+k^2-l^2+i+k+l-\sigma_r} p^{i+k-l+1}.$$

$$\begin{aligned}
\sum_{D_{V,W}^1(t_1, t_2)} \mathbb{P}_p(\{\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}\} \cap \mathcal{I}_W) &\leq \sum_{D_{V,W}^1(t_1, t_2)} O\left(n^{4i-2\sigma_t+i_1^2+i_2^2+k^2+3k+i_1+i_2-\sigma_s} p^{i_1+i_2+k+3}\right) \\
&= O(Q(t, s, r)).
\end{aligned}$$

This is apparent when $i_1 + i_2 = i - 1$. The expression is also decreasing in both i_1 and i_2 for $i_1 \leq j$ and $i_2 \leq j$, so it still holds if $i_1 + i_2 > i - 1$.

A similar argument shows

$$(43) \quad \sum_{D_{V,W}^2(t_1, t_2, r_2)} \mathbb{P}_p(\{\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}\} \cap \mathcal{I}_W) \\ \leq \sum_{D_{V,W}^2(t_1, t_2, r_2)} O\left(n^{t-t_1+r-r_2+i_1^2+3i_1+i_2^2+k^2-l_2^2+i_2+k+l_2-\sigma_{r_2}} p^{i_1+i_2+k-l_2+2}\right).$$

Factoring out $Q(t, s, r)$ from Equation 43 gives

$$Q(t, s, r) \sum_{D_{V,W}^2(t_1, t_2, r_2)} O\left(n^{-i^2+i+i_1^2+i_2^2+i_1+i_2-l_2^2+l^2+2l-l_2-k-\sigma_t} p^{-i+i_1+i_2+l-l_2}\right) \\ = O(Q(t, s, r)).$$

For $(V_1, V_2) \in D_{V,W}^3(t_1, t_2, t_3, t_4)$ it is a little more involved. First note that

$$\mathbb{P}_p(\{\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}\} \cap \mathcal{I}_W) \leq \mathbb{P}_p(\mathcal{I}_{V_1} \cap \mathcal{I}_W) \mathbb{P}_p(\mathcal{I}_{U_2 \rightarrow V_2}).$$

Then using Lemma 4.1 we have

$$\sum_{D_{V,W}^3(t_1, t_2, r_1, r_2)} \mathbb{P}_p(\{\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}\} \cap \mathcal{I}_W) \\ \leq O\left(n^{2r-r_1-r_2+i_1^2+k^2-l_1^2+i_1+k+l_1-\sigma_{r_1}+i_2^2-l_2^2+E(t_2, r_2)} p^{i_1+k-l_1+1+i_2-l_2+e(t_2, r_2)}\right) \\ = O(Q(t, s, r)) R_6(t, r, t_1, r_1, t_2, r_2)$$

where

$$R_6(t, r, t_1, r_1, t_2, r_2) = \\ n^{2r+\sigma_r-r_1-r_2+i_2^2+i_2^2-l_1^2-l_2^2-i^2+l^2+i_1+l_1-\sigma_{r_1}-i-l+E(t_2, r_2)} p^{i_1+i_2-i-l_1-l_2+l+e(t_2, r_2)}.$$

We see that

$$R_6(t, r, t_1, r_1, t_2, r_2) = \\ R_5(t, r, t_1, r_1, t_2, r_2) n^{i_1+l_1+\sigma_r-\sigma_{r_1}-i-l-E(t_1, r_1)+E(t, r)} p^{e(t, r)-e(t_1, r_1)} \\ = O(1)$$

by a closer inspection of $R_5(t, r, t_1, r_1, t_2, r_2)$ from the fifth part of Lemma 4.1. One does need to worry if $e(t, r)$ is 0 and $e(t_1, r_1) = 1$, but the broken symmetry allows us to switch the upper bound to

$$R_5(t, r, t_1, r_1, t_2, r_2) n^{i_2+l_2+\sigma_r-\sigma_{r_2}-i-l-E(t_2, r_2)+E(t, r)} p^{e(t, r)-e(t_2, r_2)}.$$

If both $e(t_1, r_1) = e(t_2, r_2) = 1$ and $e(t, r) = 0$ then we note that $R_5(t, r, t_1, r_1, t_2, r_2) = O(n^{2i_1} p)$. ■

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